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IDEMPOTENT FUNCTIONS ON TOPOLOGICAL DYNAMICS

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Abstract. In this article, we deal with notion of idempotent in dynamical systems and prove that both closure function and orbital function are idempotent functions on the systems.

1. Introduction

The notion of idempotent provides simplicity to analyze dynamical systems. Conley [6] introduced a lot of notions on topological dynamics on compact metric spaces. Multi-valued dynamical systems appear to be rather suitable for describing the global behavior of processes in optimal control dynamics and economic dynamics and are used to describe multi-valued differential equations and appear naturally in control systems. Akin [1] and McGehee [7] obtained many results in the multi-valued dynamics. Chu *et al.* [2, 3, 5, 8] studied with several notions in multi-valued dynamical systems.

In this paper, we deal with notions on multi-valued dynamical systems. We focus on closure functions and orbital functions of multi-valued functions on the systems. It is worth while studying iterations of the multi-valued functions to get much more information for the systems. We show that the above two functions have idempotent property.

From now on, let Γ be a multi-valued function from a Hausdorff space X.

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To avoid ambiguity of the definitions of the functions, we define canonically the new mapping from the power set of X to the same set as follows,

$$\Gamma(A) := \cup_{x\in A}\Gamma(x),$$

here $A \subseteq X$. So we develop the definition of the composition $\Gamma^2 = \Gamma\circ\Gamma$ given by $\Gamma^2(x) := \Gamma(\Gamma(x)) = \cup_{y\in\Gamma(x)}\Gamma(y)$, and so we define the iteration $\Gamma^n : X \to 2^X$ inductively by $\Gamma^1(x) := \Gamma(x)$ and $\Gamma^n(x) := \Gamma(\Gamma^{n-1}(x))$. The *orbit* $Orb_\Gamma(x)$ of a point x in X for the function Γ is expressed by the union of the iterations of the point, that is, $Orb_\Gamma(x) := \cup_{n\in\mathbb{Z}_+}\Gamma^n(x)$.

Let $\mathcal{P}$ be the set of all functions from X to its power set 2^X and let $\Gamma \in \mathcal{P}$. We define the mappings D and S from the set $\mathcal{P}$ to itself given by, for every $x \in X$,

$$(D\Gamma)(x) = \cap_{U\in N(x)}\overline{\Gamma(U)} \quad \text{and} \quad (S\Gamma)(x) = \cup_{n=1}^{\infty}\Gamma^n(x).$$

Here, $N(x)$ is the set of all neighborhoods of x. We call D a *closure funtion* for Γ and S an *orbital function* for Γ defined on $\mathcal{P}$.

Remark 1.1. [4] Let X be a locally compact metric space. From the metric on the space, it is convenient to study theories about the functions on dynamical systems. Let $\Gamma : X \to 2^X$ be a multi-valued function and $x \in X$. Then $D\Gamma(x)$ is the set of all points $y \in X$ with the property that there exist sequences (x_n) and (y_n) in X with $y_n \in \Gamma(x_n)$ such that $x_n \to x$, $y_n \to y$. Furthermore, $S\Gamma(x)$ is the set of all points $y \in X$ such that there is a finite subset $\{x_1, \cdots, x_k\}$ of X $(k \in \mathbb{Z}_+)$ with the properties that $x_1 = x, x_k = y$ and $x_{i+1} \in \Gamma(x_i)$, $i = 1, \cdots, k-1$.

2. Idempotent functions on dynamical systems

In this section, we prove the new functions have idempotent property, that is, the iterations of the functions are just the original functions.

Theorem 2.1. *A closure function D is idempotent, that is, $D^2 = D$.*

Proof. We first show that $D^2\Gamma(x)$ is contained in $D\Gamma(x)$. From the definition of the mapping, we directly get the equalities

$$\begin{aligned} D^2\Gamma(x) &= (D(D\Gamma))(x) \\ &= \cap_{U\in N(x)}\overline{D\Gamma(U)} \\ &= \cap_{U\in N(x)}\overline{\cup_{y\in U}D\Gamma(y)} \\ &= \cap_{U\in N(x)}\overline{\cup_{y\in U}\cap_{V\in N(y)}\overline{\Gamma(V)}}. \end{aligned}$$

Note that, for each $y \in U$, we obtain the following inclusion

$$\cap_{V\in N(y)}\overline{\Gamma(V)} \subseteq \overline{\Gamma(U)}.$$

Thus we have that

$$\cup_{y\in U} \cap_{V\in N(y)} \overline{\Gamma(V)} \subseteq \overline{\Gamma(U)},$$

and so

$$\cap_{U\in N(x)}\overline{\cup_{y\in U} \cap_{V\in N(y)} \overline{\Gamma(V)}} \quad \subseteq \quad \cap_{U\in N(x)}\overline{\Gamma(U)} = D\Gamma(x).$$

Conversely, we look at the opposite direction of the proof to get the equality. For every neighborhood V of y, we get $\Gamma(y) \subseteq \cap_{V\in N(y)}\overline{\Gamma(V)}$. By the definitions, we obtain that

$$\overline{\Gamma(U)} = \overline{\cup_{y\in U}\Gamma(y)} \subseteq \overline{\cup_{y\in U} \cap_{V\in N(y)} \overline{\Gamma(V)}},$$

so we conclude that

$$D\Gamma(x) = \cap_{U\in N(x)}\overline{\Gamma(U)} \subseteq \cap_{U\in N(x)}\overline{\cup_{y\in U} \cap_{V\in N(y)} \overline{\Gamma(V)}} = D^2\Gamma(x).$$

Then we have the first equality of this proof. □

In the next remark, we check precisely that the orbital function has the interesting property.

Remark 2.2. The orbital function S is idempotent, that is, $S^2 = S$. Indeed, from the definition of the functions, we get that

$$\begin{aligned} S^2\Gamma(x) &= (S(S\Gamma))(x) \\ &= \cup_{n=1}^{\infty}(S\Gamma)^n(x) \\ &= (S\Gamma)(x) \cup (S\Gamma)^2(x) \cup \cdots , \end{aligned}$$

for every $x \in X$. So we have that $(S^2\Gamma)(x) \supseteq (S\Gamma)(x)$.

Now we prove another inclusion $S(S\Gamma)(x) \subseteq (S\Gamma)(x)$. We first obtain that

$$
\begin{aligned}
(S\Gamma)^2(x) &= (S\Gamma)((S\Gamma)(x)) \\
&= \cup_{k=1}^{\infty}\Gamma^k(S\Gamma(x)) \\
&= \cup_{y\in S\Gamma(x)} \cup_{k=1}^{\infty} \Gamma^k(y) \\
&= \cup_{y\in \cup_{n=1}^{\infty}\Gamma^n(x)} \cup_{k=1}^{\infty} \Gamma^k(y) \\
&= \cup_{n=1}^{\infty} \cup_{y\in\Gamma^n(x)} \cup_{k=1}^{\infty}\Gamma^k(y) \\
&= \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} \cup_{y\in\Gamma^n(x)}\Gamma^k(y) \\
&= \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} \Gamma^k(\Gamma^n(x)) \\
&\subseteq \cup_{n=1}^{\infty}\Gamma^n(x) \\
&= S\Gamma(x).
\end{aligned}
$$

Thus we have $(S\Gamma)^2(x) \subseteq S\Gamma(x)$. Using the induction, we assume that $(S\Gamma)^l(x) \subseteq (S\Gamma)(x)$ for every positive integer $l \leq m$. Then, by the properties of the mappings, we have that

$$
\begin{aligned}
(S\Gamma)^{m+1}(x) &= (S\Gamma)((S\Gamma)^m(x)) \\
&\subseteq (S\Gamma)((S\Gamma)(x)) \\
&\subseteq S\Gamma(x).
\end{aligned}
$$

So $(S\Gamma)^n(x) \subseteq (S\Gamma)(x)$, for every positive integer n. Then we get

$$\cup_{n=1}^{\infty}(S\Gamma)^n(x) \subseteq (S\Gamma)(x) \text{ and so } S(S\Gamma)(x) \subseteq S\Gamma(x) \text{ for all } x.$$

Hence we have $S^2\Gamma = S\Gamma$, as desired.

Since images of a closure function and an orbital function are both in $\mathcal{P}$, we consider the notions of cluster function and transitivity in $\mathcal{P}$. A multi-valued function $\Gamma : X \to 2^X$ is a *cluster function* if $D\Gamma = \Gamma$. The function Γ is *transitive* provided $S\Gamma = \Gamma$. It is obvious that Γ is transitive if $\Gamma^2 = \Gamma$.

Corollary 2.3. *Let $\Gamma : X \to 2^X$ be a multi-valued function. Then the image $D\Gamma$ of Γ for the closure function is cluster and the image $S\Gamma$ of Γ for the orbital function is transitive.*

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